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L^p-convergence of greedy algorithm by generalized Walsh system

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ABSTRACT

The main goals of this paper are to consider a problem of L^p-convergence $(p > 2)$ of greedy algorithm with respect to generalized Walsh system.

1. Introduction

Let $a \geqslant 2$ be a fixed integer and $\omega_a = e^{\frac{2\pi i}{a}}$. Recall the following definitions (see [1]).

The *Rademacher system of order a* is defined inductively as follows. For $n = 0$ let

$$\varphi_0(x) = \omega_a^k \quad \text{if } x \in \left[\frac{k}{a}, \frac{k+1}{a}\right), \ k = 0, 1, \dots, a-1,$$

and for $n \geqslant 1$ let

$$\varphi_n(x+1) = \varphi_n(x) = \varphi_0\left(a^n x\right).$$

The *generalized Walsh system of order a* is defined by

$$\psi_0(x) = 1,$$

and if $n = \alpha_1 a^{n_1} + \cdots + \alpha_s a^{n_s}$, where $n_1 > \cdots > n_s$, $0 \leqslant \alpha_j < a$, $j = 1, 2, \dots, s$ then

$$\psi_n(x) = \varphi_{n_1}^{\alpha_1}(x) \cdot \dots \cdot \varphi_{n_s}^{\alpha_s}(x).$$

We denote the generalized Walsh system of order a by Ψ_a. Not that Ψ_2 is the classical Walsh system. The basic properties of the generalized Walsh system of order a have been obtained by H.E. Chrestenson, J. Fine, C. Vatari, W. Young, N. Vilenkin and others (see [1–7]). Next we list some properties of Ψ_a, which will be useful later.

- Each n-th Rademacher function has period a^{-n} and

$$\varphi_n(x) = const \in \Omega_a = \left\{1, \omega_a, \omega_a^2, \dots, \omega_a^{a-1}\right\}, \tag{1}$$

if $x \in \Delta_{n+1}^{(k)} = [\frac{k}{a^{n+1}}, \frac{k+1}{a^{n+1}})$, $k = 0, \dots, a^{n+1} - 1$, $n = 1, 2, \dots$.

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- $(\varphi_n(x))^k = (\varphi_n(x))^m$, $\forall n, k \in \mathcal{N}$, and $m = k \ (mod\ a)$.
- $\psi_n(x)$ is a finite product of Rademacher functions with values in Ω_a.
- $\psi_{a^k+j}(x) = \varphi_k(x) \cdot \psi_j(x)$, if $0 \leqslant j \leqslant a^k - 1$.
- Ψ_a, $a \geqslant 2$ is a complete orthonormal system in $L^2[0,1)$ (see [2, pp. 5, 30]) and it is a basis in $L^p[0,1]$ for $p > 1$ (see [7]).

In this paper we consider L^p-convergence of the greedy algorithm with respect to the system Ψ_a for $p > 2$.

Let X be a Banach space with a norm $\|\cdot\| = \|\cdot\|_X$ and a basis $\Phi = \{\phi_k\}_{k=1}^{\infty}$, $\|\phi_k\|_X = 1$, $k = 1, 2, \ldots$. For a function $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} a_k(f)\phi_k.$$

Definition. The *m-th greedy approximant of $f \in X$ with respect to Φ* is defined by

$$G_m(f, \phi) = \sum_{k \in \Lambda} a_k(f)\phi_k,$$

where $\Lambda \subset \{1, 2, \ldots\}$ is an arbitrary set of cardinality m such that

$$\left|a_n(f)\right| \geqslant \left|a_k(f)\right|, \quad \text{whenever } n \in \Lambda,\ k \notin \Lambda.$$

In particular we will say that the greedy approximant of $f \in L^p[0,1]$, $p \geqslant 0$ converges with regard to the Ψ_a, if the sequence $G_m(x, f)$ converges to $f(t)$ in L^p norm.

Recently there has been a surge of interest in the theory of greedy approximations (see [8–16]). T.W. Körner constructed in [10] an L^2 function (then a continuous function) whose greedy algorithm with respect to trigonometric systems diverges almost everywhere. V.N. Temlyakov in [11] gave an example of a function $f \in L^p$, $p \in [1, 2)$ (resp. $p > 2$), whose greedy algorithm with respect to trigonometric systems diverges in measure (resp. in L^p, $p > 2$). R. Gribonval and M. Nielsen proved that for any $p > 1$ there is a function $f \in L^p[0,1)$ whose greedy algorithm with respect to Ψ_a, $a \geqslant 2$ systems diverges in $L^p[0,1]$ (see [13]). Note also that for $L^1[0,1)$ the analogous result was proved in [15,16].

In this paper we are concerned with the following question. *Is it possible to modify $f \in L^p$ on a small set so that the greedy algorithm of the modified function converges in L^p*? We will prove that this is indeed the case.

Theorem 1. *Let $p > 2$. For every $\varepsilon > 0$ and $f \in L^p[0,1]$ there is a function $g \in L^p[0,1]$, such that $|\{x \in [0,1];\ g \neq f\}| < \varepsilon$ and the greedy algorithm of g converges in L^p.*

This result will follow from the next, more general theorem, which may be of independent interest.

Theorem 2. *Let $p > 2$. For any $\varepsilon > 0$ and $f \in L^p[0,1]$ there is a function $g \in L^p[0,1]$, such that $|\{x \in [0,1]:\ g \neq f\}| < \varepsilon$, such that the sequence $\{|c_k(g)|,\ k \in \operatorname{spec}(g)\}$, is monotonically decreasing.*

Recall, that the spectrum of f (denoted by $\operatorname{spec}(f)$) is the support of $\{c_k(f)\}$, i.e. the set of integers k for which $c_k(f) \neq 0$.

The idea of modifying a function in order to improve its properties dates back to Luzin (see [17]) and it was substantially developed later on. For instance in 1939, Men'shov [18] proved the following fundamental theorem.

Theorem *(Men'shov's C-strong property). Let $f(x)$ be an a.e. finite measurable function on $[0, 2\pi]$. Then for each $\epsilon > 0$ there is a continuous function $g(x)$ coinciding with $f(x)$ on a subset E of measure $|E| > 2\pi - \varepsilon$ such that its Fourier series with respect to the trigonometric system converges uniformly on $[0, 2\pi]$.*

In [20] it was proved that for any $\varepsilon \in (0,1)$, $p \geqslant 1$ and for any function $f \in L^p[0,1]$ there is a function $g \in L^{\infty}[0,1]$, $|\{x \in [0,1];\ g \neq f\}| < \epsilon$, such that the greedy algorithm of g with respect to the classical Walsh Ψ_2 system converges uniformly on $[0,1)$. For further results in this direction we refer the reader to the papers [19–23].

2. Basic lemmas

For $m = 1, 2, \ldots$ and $1 \leqslant k \leqslant a^m$ we put $\Delta_m^{(k)} = [\frac{k-1}{a^m}, \frac{k}{a^m})$ and consider the following function

$$I_m^{(k)}(x) = \begin{cases} 1, & \text{if } x \in [0,1) \setminus \Delta_m^{(k)}, \\ 1 - a^m, & \text{if } x \in \Delta_m^{(k)}, \end{cases} \tag{2}$$

and periodically extend these functions on R^1 with period 1.

By $\chi_E(x)$ we denote the characteristic function of the set E, i.e.

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases} \tag{3}$$

Then, clearly

$$I_m^{(k)}(x) = \psi_0(x) - a^m \cdot \chi_{\Delta_m^{(k)}}(x), \tag{4}$$

and let for the natural numbers $m \geqslant 1$ and $1 \leqslant i \leqslant a^m$

$$a_i(\chi_{\Delta_m^{(k)}}) = \int_0^1 \chi_{\Delta_m^{(k)}}(x) \cdot \overline{\psi_i(x)}\, dx = \mathcal{A} \cdot \frac{1}{a^m}, \quad 0 \leqslant i < a^m, \tag{5}$$

$$b_i\big(I_m^{(k)}\big) = \int_0^1 I_m^{(k)}(x) \cdot \overline{\psi_i(x)}\, dx = \begin{cases} 0, & \text{if } i = 0 \text{ and } i \geqslant a^m, \\ -\mathcal{A}, & \text{if } 1 \leqslant i < a^m, \end{cases} \tag{6}$$

where $\mathcal{A} = const \in \Omega_a$ and $|\mathcal{A}| = 1$.

Hence

$$\chi_{\Delta_m^{(k)}}(x) = \sum_{i=0}^{a^k-1} b_i(\chi_{\Delta_m^{(k)}})\psi_i(x), \tag{7}$$

$$I_m^{(k)}(x) = \sum_{i=1}^{a^k-1} a_i\big(I_m^{(k)}\big)\psi_i(x). \tag{8}$$

Lemma 1. *For any given numbers $\gamma \neq 0$, $N_0 > 1$, $\varepsilon \in (0,1)$, $p > 2$ and a-dyadic interval $\Delta = \Delta_m^{(k)} = [\frac{k-1}{a^m}, \frac{k}{a^m})$, $i = 1, \dots, a^m$ there exist a measurable set $E \subset \Delta$ and a polynomial $P(x)$ by Ψ_a of the form*

$$P(x) = \sum_{k=N_0}^{N} c_k \psi_k(x)$$

which satisfy the conditions:

(1) *the coefficients $\{c_k\}_{k=N_0}^N$ are 0 or $-\mathcal{K} \cdot \gamma \cdot |\Delta|$,*

where $\mathcal{K} = const \in \Omega_a$, $|\mathcal{K}| = 1$,

(2) $|E| > (1-\varepsilon) \cdot |\Delta|$,

(3) $P(x) = \begin{cases} \gamma, & \textit{if } x \in E; \\ 0, & \textit{if } x \notin \Delta, \end{cases}$

(4) $\|P\|_p = \left[\int_0^1 |P(x)|^p\, dx \right]^{\frac{1}{p}} < \dfrac{a^2}{\varepsilon^{\frac{1}{q}}} \cdot |\gamma| \cdot |\Delta|^{\frac{1}{p}}$.

Proof. Let

$$\nu_0 = \left[\log_a \frac{1}{\varepsilon}\right] + 1; \qquad s = [\log_a N_0] + m. \tag{9}$$

We define the numbers c_n, a_i and b_j and a polynomial $P(x)$ and in the following form:

$$P(x) = \gamma \cdot \chi_{\Delta_m^{(k)}}(x) \cdot I_{\nu_0}^{(1)}\big(2^s x\big), \quad x \in [0,1], \tag{10}$$

$$c_n = c_n(P) = \int_0^1 P(x)\varphi_n(x)\, dx, \quad \forall n \geqslant 0, \tag{11}$$

$$a_i = a_i(\chi_{\Delta_m^{(k)}}), \quad 0 \leqslant i < a^m, \qquad b_j = b_j\big(I_{\nu_0}^{(1)}\big), \quad 1 \leqslant j < a^{\nu_0}. \tag{12}$$

Taking into consideration the following equation

$$\psi_i(x)\cdot\psi_j\big(a^s x\big)=\psi_{j\cdot a^s+i}(x),\quad \text{if } 0\leqslant i,\ j<a^s,$$

and having the following relations (1), (5)–(8) and (10)–(12), we obtain that the polynomial $P(x)$ has the following form:

$$\begin{aligned}P(x)&=\gamma\cdot\sum_{i=0}^{a^m-1}a_i\psi_i(x)\cdot\sum_{j=1}^{a^{\nu_0}-1}b_j\psi_j\big(a^s x\big)\\&=\gamma\cdot\sum_{j=1}^{a^{\nu_0}-1}b_j\cdot\sum_{i=0}^{a^m-1}a_i\psi_{j\cdot 2^s+i}(x)=\sum_{k=N_0}^{N}c_k\psi_k(x),\end{aligned}$$

where

$$c_k=c_k(P)=\begin{cases}-\mathcal{K}\cdot\frac{\gamma}{a^m}\ \text{or}\ 0, & \text{if } k\in[N_0,N],\\ 0, & \text{if } k\notin[N_0,N],\end{cases}$$

where

$$\mathcal{K}\in\Omega_a,\quad |\mathcal{K}|=1,\quad N=a^{s+\nu_0}+a^m-a^s-1.$$

Set

$$E=\big\{x;\ P(x)=\gamma\big\}.$$

Clearly that (see (2) and (10)),

$$\begin{aligned}&|E|=a^{-m}\big(1-a^{-\nu_0}\big)>(1-\epsilon)|\Delta|,\\&P(x)=\begin{cases}\gamma, & \text{if } x\in E,\\ \gamma\big(1-a^{\nu_0}\big), & \text{if } x\in\Delta\setminus E,\\ 0, & \text{if } x\notin\Delta.\end{cases}\end{aligned}$$

Thus, for $p\geqslant 2$ and $\frac{1}{p}+\frac{1}{q}=1$

$$\left(\int_0^1\big|P(x)\big|^p\,dx\right)^{\frac{1}{p}}\leqslant a^2\cdot|\gamma|\cdot\varepsilon^{-\frac{1}{q}}\cdot|\Delta|^{\frac{1}{p}}.$$

Lemma 1 is proved. □

Lemma 2. *For any given numbers $p>2$, $N_0>1$, $\epsilon\in(0,1)$ and $\delta>0$ and a polynomial $f(x)$ by Ψ_a system there exist a measurable set $E\subset[0,1]$ and a polynomial by Ψ_a of the form*

$$P(x)=\sum_{k=N_0}^{N}c_k\psi_k(x),$$

satisfying the following conditions:

(1) $0\leqslant|a_k|<\delta$ *and non-zero coefficients* $\big\{|a_k|\big\}_{k=m_0}^{N}$ *are decreasing*,

(2) $|E|>1-\epsilon$,

(3) $P(x)=f(x)$, *for all* $x\in E$,

(4) $\|P\|_p<\dfrac{a^2}{\epsilon^{\frac{1}{q}}}\|f\|_p\quad\left(\dfrac{1}{p}+\dfrac{1}{q}=1\right)$.

Proof. Let

$$f(x)=\sum_{k=0}^{M}b_k\psi_k(x)=\sum_{s=1}^{m}\gamma_s\cdot\chi_{\Delta_s}(x),\qquad\sum_{s=1}^{m}|\Delta_s|=1,\tag{13}$$

where Δ_s are a-dyadic intervals of the form $\Delta_n^{(k)}$, $k=1,2,\dots,a^n$. Without loss of generality, one may assume that

$$0<|\gamma_1||\Delta_1|<\cdots<|\gamma_s||\Delta_s|<\cdots<|\gamma_m||\Delta_m|<\delta. \tag{14}$$

Applying Lemma 1 successively, we can find the sets $E_s\subset[0,1]$ and the polynomials

$$P_s(x)=\sum_{j=N_{s-1}}^{N_s-1}c_j\psi_j(x),$$
$$c_j=0 \quad \text{or} \quad -\mathcal{K}\cdot\gamma_s\cdot|\Delta_s| \quad \text{if } j\in[N_{s-1},N_s),\ s=1,2,\dots,m, \tag{15}$$

which satisfy the following conditions:

$$|E_s|>(1-\epsilon)\cdot|\Delta_s|, \tag{16}$$
$$P_s(x)=\begin{cases}\gamma_s, & \text{if } x\in E_s,\\ 0, & \text{if } x\notin\Delta_s,\end{cases} \tag{17}$$
$$\|P_s\|_p=\left(\int_0^1\left|P_s(x)\right|^p dx\right)^{1/p}<\frac{a^2\cdot|\gamma_s|}{\epsilon^{\frac{1}{q}}}\cdot|\Delta_s|^{1/p}. \tag{18}$$

We define

$$P(x)=\sum_{s=1}^m P_s(x)=\sum_{k=N_0}^N c_k\psi_k,\quad N=N_{m-1}, \tag{19}$$
$$E=\bigcup_{s=1}^m E_s. \tag{20}$$

From (13)–(17), (19) and (20) we have

$$0\leqslant|a_k|<\delta \text{ and the non-zero coefficients } \left\{|c_k|\right\}_{k=N_0}^N \text{ are decreasing,}$$
$$P(x)=f(x),\quad \text{for } x\in E, \tag{21}$$
$$|E|>1-\epsilon. \tag{22}$$

Taking into account (13), (18), (19) and $\frac{1}{p}+\frac{1}{q}=1$ we obtain

$$\int_0^1\left|P(x)\right|^p dx=\sum_{\nu=1}^m\int_{\Delta_\nu}\left|\sum_{s=1}^m P_s(x)\right|^p dx \tag{23}$$
$$=\sum_{\nu=1}^m\int_{\Delta_\nu}\left|P_\nu(x)\right|^p dx\leqslant\sum_{\nu=1}^m\frac{a^2\cdot|\gamma_\nu|}{\epsilon^{\frac{p}{q}}}\cdot|\Delta_\nu|\leqslant a^{2p}\cdot\frac{\int_0^1|f(x)|^p dx}{\epsilon^{p-1}}. \tag{24}$$

Lemma 2 is proved. □

3. Proof of Theorem 2

Let given any function $f(x)\in L^p[0,1]$, $p>2$ and any $\varepsilon\in(0,1)$. It is easy to see that we can find a sequence $\{f_n(x)\}_{n=1}^\infty$ of polynomials in the Ψ_a system so that

$$\lim_{N\to\infty}\left\|\sum_{n=1}^N f_n(x)-f(x)\right\|_p dx=0,$$
$$\left\|f_n(x)\right\|_p\leqslant\varepsilon^{\frac{1}{q}}\cdot a^{-2n},\quad n\geqslant 2\left(\frac{1}{p}+\frac{1}{q}=1\right).$$

Applying repeatedly Lemma 2, we can find a sequences of sets $\{E_n\}_{n=1}^\infty$ and polynomials by Ψ_a of the form

$$P_n(x)=\sum_{k=M_{n-1}}^{M_n-1}c_k\psi_k(x),\quad n\geqslant1,\ M_n\nearrow,$$

which satisfy the following conditions for all $n \geqslant 1$:

$$P_n(x) = f_n(x), \quad \text{for } x \in E_n,$$
$$|E_n| > 1 - \varepsilon \cdot a^{-n},$$
$$\|P_n\|_p \leqslant \frac{a^2}{\varepsilon^{\frac{1}{q}}} \cdot a^{\frac{n}{q}} \cdot \|f_n\|_p,$$
$$|c_k| < |c_{k+1}| < 2^{-n}, \quad \text{for all } k \in [M_{n-1}; M_n).$$

We set

$$g(x) = \sum_{n=1}^{\infty} P_n(x), \qquad E = \bigcap_{n=1}^{\infty} E_n.$$

It is easy to see that

$$\bigl\{|c_k(g)|,\ k \in \operatorname{spec}(g)\bigr\} \text{ is monotonically decreasing},$$
$$g(x) \in L^p[0,1),$$
$$g(x) = f(x), \quad \text{for } x \in E,\ |E| > 1 - \varepsilon.$$

Theorem 2 is proved.

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